Conformal Maps of Nonsmooth Surfaces and Their Applications

Vladimir M. Miklyukov

Exlibris Corporation, Philadelphia
2008

Library of Congress Control Number: 2008903546
ISBN: Hardcover 978-1-4363-3693-2
Softcover 978-1-4363-3692-5

This book was printed in the United States of America.

To order additional copies of this book, contact:
Xlibris Corporation
1-888-795-4274
www.Xlibris.com
Orders@Xlibris.com

Preface

Astonishingly, under the large number of books on conformal maps between plane domains, the corresponding theory for surfaces is not represented with a connected account up to now.

The aim of this book is the introduction to the theory of conformal maps between non-regular surfaces. We are restricted to examination of locally Lipschitz surfaces. Apparently, such class is minimally necessary in generalizations and reasonably enough for applications.

Below following problems are touched upon: the existence and uniqueness of maps, the boundary behavior of maps and prime ends of surfaces, theorems of the Ahlfors and Warschawski type, applications in a problem of the gas dynamics equation and qualitative questions of the theory of minimal surfaces.

Remark that the considered class of problems does not require applications of complex variables. Thus we can courageously offer another title of this book: "Conformal map without complex variables". Nevertheless, in such places where the application of complex variables is justified by concepts of convenient, we use the language of complex variables. The author attempted to make this presentation maximally simple such that it will be accessible for everybody including young mathematicians started professional work in this domain.

In the book base, there are lectures written by the author for masters of the department of the mathematical analysis and the function theory of Volgograd State University in the 2004/05 academic year. Therefore, as a rule, in the case of the necessary choice between the account of the result in the maximal generality and the showing of a method of its receipt, we preferred the last and refered to find the generality at original articles.[1]

We formulate a series of unsolved problems for beginners.

The author hopes that the book will be useful to readers, interested in conformal mappings and their applications.

Vladimir Michaelovich Miklyukov
Laboratory "Superslow processes",
Volgograd State University,
University avenue 100, Volgograd 400062, RUSSIA
E-mail: `miklyuk@mail.ru`

[1]Rephrasing the well-known aphorism: 'We do not sell fish, we sell fishing-tackles'.

Contents

Chapter 10

Solutions Close to a Boundary

We study generalized solutions of minimal surface type equation. We prove that every solution has on the boundary no more than a countable number of jumps. In particular, every solution, defined in the disc exterior, is continuously extendable up to the boundary circle, excepting possibly a countable set of points. For Fatou's type theorems about angular boundary values, see [109], [83], [115], [100], [120].

10.1 Main Results

Let $D \subset \mathbb{R}^2$ be a domain and let $e \subset D$ be a set of zero linear Hausdorff measure.

Let

$$A = (A_1(x, \xi), A_2(x, \xi)) : (D \setminus e) \times \mathbb{R}^2 \to \mathbb{R}^2$$

be a continuous vector function. Suppose that, for every point $x = (x_1, x_2) \in D \setminus e$ and every $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, the following structure conditions hold:

$$\nu_1 \frac{|\xi|^2}{\sqrt{1 + |\xi|^2}} \leq \sum_{i=1}^{2} \xi_i \, A_i(x, \xi), \tag{10.1}$$

$$|A(x, \xi)| \leq \nu_2(x) \tag{10.2}$$

where ν_1 is a positive constant and $\nu_2(x)$ is a positive continuous function.

Below we will assume that $\nu_2(x)$ satisfies

$$\underline{\lim}_{\{D_n\}} \int\limits_{\partial D_n} \nu_2(x)\,|dx| = \nu_2^* < \infty \tag{10.3}$$

where the lower limit is taken over all sequences $\{D_n\}$ of subdomains of D with rectifiable boundaries for which $\overline{D_n} \subset D_{n+1}$, $\cup_{n=1}^{\infty} \overline{D}_n = D$. (Here and below, the symbol $\overline{H}$ means the closure of the set $H \subset \mathbb{R}^2$ with respect to the topology $\mathbb{R}^2$.)

Consider the equation

$$\sum_{i=1}^{2} \frac{d}{dx_i} A_i(x, \nabla f) = 0\,. \tag{10.4}$$

As above in Chapter 8, we use the following definition of the generalized solution. Denote by $D_b(f)$ the subset of D, at every point of which the function f is not differentiable. By a *generalized solution* of (10.4), we call a locally Lipschitz function f with the following property. For every bounded subdomain Δ, $\overline{\Delta} \subset D$, with the rectifiable boundary $\partial\Delta$, $\mathrm{mes}_1\,(\partial\Delta \cap D_b(f)) = 0$, and a function $\varphi \in \mathrm{Lip}\,\overline{\Delta}$, the following equality holds

$$\int\limits_{\partial\Delta} \varphi \sum_{i=1}^{2} A_i(x, \nabla f)\, n_i\,|dx| = \iint\limits_{\Delta} \sum_{i=1}^{2} \varphi'_{x_i}\, A_i(x, \nabla f)\, dx_1\, dx_2\,. \tag{10.5}$$

Here and below, $\overline{n} = (n_1(x), n_2(x))$ is a unit vector of the exterior normal to the boundary $\partial\Delta$.

The set of the discontinuity points of the vector function A has zero linear Hausdorff measure, and hence, the contour integral in (10.5) exists.

Exercise 10.6 Prove (or refute !) that classes of generalization solutions of minimal surface type equations (9.1), (9.2), (9.3), introduced here and in Chapter 9, coincide.

□

Let f be a continuous function, defined in the domain $D \subset \mathbb{R}^2$ with the rectifiable boundary. The function f has a *finite (or infinite) angular boundary value* α at $a = (a_1, a_2) \in \partial D$ if

$$\lim_{x \to a} f(x) = \alpha$$

along every angle C with a vertex at a lying inside D.

The following theorem is a version of Fatou's theorem [147, Chapter I, §**5**]. *Every bounded, harmonic in a unit disc B function f has angular boundary values a.e. on the circle ∂B.*

Moreover, there are examples of unbounded harmonic functions which have no angular boundary values on sets $H \subset \partial B$ with linear measure $\mathrm{mes}_1 \mathrm{H} > 0$ [28, Chapter 2, §**10**].

Problem. (J.C.C. Nitsche [137, Chapter VII, n. **4**]) Do valid Fatou's type theorems exist for solutions of (2.20)?

The following result was obtained in [115].

(A) *Every solution of the minimal surface equation* (2.20) *has finite or infinite angular boundary values a.e. on ∂B.*

Let's emphasize that here we do not suppose additional restrictions for solutions.

At the same time, it is necessary to remark that the behavior of solutions of (2.20) depends on the specific structure of the domain where these solutions are defined. Namely, for the solutions, defined in the exterior of a disc B, the following statement holds [115].

(B) *Every solution of the minimal surface equation* (2.20), *defined over $\mathbb{R}^2 \setminus \overline{B}$, is continuously extended a.e. to the boundary, i.e. a finite limit*

$$\lim_{x \to a} f(x) = \alpha, \quad x \in \mathbb{R}^2 \setminus \overline{B}$$

exists a.e. on the circle ∂B.

Theorem 10.7 *Let $D \subset \mathbb{R}^2$ be a domain with the Jordan rectifiable boundary. Every generalized solution of the equation* (10.4) *with the structure conditions* (10.1), (10.2), (10.3) *has finite or infinite angular boundary values a.e. on ∂D.*

On generalizations, see [100], [120].

Example 10.8 A solution of (2.20) can have infinite values on a set of the positive linear measure. Consider the Scherk surface

$$f(x_1, x_2) = \log \frac{\cos x_2}{\cos x_1},$$

defined over the square

$$Q = \left\{ (x_1, x_2) : -\frac{\pi}{2} < x_i < \frac{\pi}{2} \quad (i = 1, 2) \right\}.$$

This surface is minimal, and the function f equals $\pm\infty$ on horizontal and vertical sides of the square boundary. In the vertices of the square, the function $f|_{\partial Q}$ has jumps.

□

Let $D \subset \mathbb{R}^2$ be a simply connected domain with the Jordan boundary ∂D and let $O \in D$ be a fixed point. Let $U \subset D$ be an open set. We denote: $[U] = \overline{U} \setminus \partial D$, $\partial' U = [U] \setminus U$.

Definition 10.9 *Let f be a continuous function defined in D. We will call $a \in \partial D$ by the point of quasi-continuity f if there is a sequence of subdomains $\{D_k\}_{k=1}^{\infty}$ of the domain D with properties:*

(α) *every $\partial' D_k$ separates a from the fixed point O;*

(β) $\cap_k [D_k] = \emptyset$; length $\partial' D_k \to 0$ *for $k \to \infty$;*

(γ) $\liminf_{k\to\infty} \operatorname{osc}(\partial' D_k,\, f) = 0$.

We will call all remaining points of ∂D by jump points of the function f.

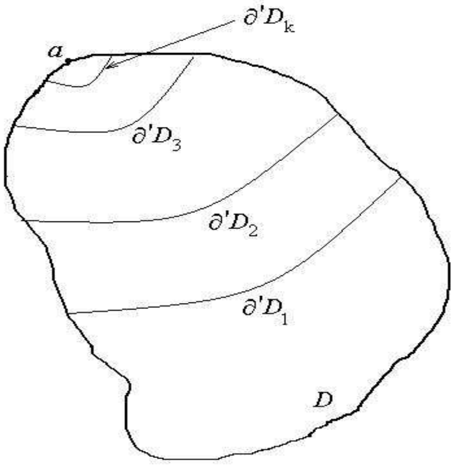

Fig. 10.1.

Define the following nonlocal characteristic of a boundary point $a \in \partial D$. We put

$$\delta(a, f, O) = \inf_{\gamma} \max\{\operatorname{osc}(\gamma, f), \operatorname{length}(\gamma)\}$$

where the infimum is taken over all arcs $\gamma \subset D$, $\overline{\gamma} \cap \partial D \neq \emptyset$, separating points a and O in D.

It is clear that f is quasi-continuous at a point $a \in \partial D$ if and only if $\delta(a, f, O) = 0$.

On the structure of solutions of (2.20) close to jump points, see Lancaster [83]. We prove that boundary quasi-continuity points of solutions f of the minimal surface type equation are typical.

Consider the set $\mathcal{H}$ of all piecewise continuous functions $h : \mathbb{R} \to \mathbb{R}$ with properties:

(i) $0 \le h(t) \le 1$ for all $t \in \mathbb{R}$;

(ii) $\int\limits_{-\infty}^{+\infty} h(t)\, dt \le h_0 < \infty, \quad h_0 = \text{const.}$

Theorem 10.10 *Let $D \subset \mathbb{R}^2$ be a simply connected domain bounded with a simple rectifiable Jordan curve ∂D, $O \in D$. Let $f : D \to \mathbb{R}$ be an arbitrary solution of an equation* (10.4) *with structure restrictions* (10.1), (10.2), (10.3).

Then for every $h \in \mathcal{H}$, the function

$$w(x) = \int\limits_{-\infty}^{f(x)} h(t)\, dt$$

is quasi-continuous at all points $a \in \partial D$, except only in a countable set.

Moreover, if $a_1, a_2, \ldots$ are jump points of f, then

$$\begin{aligned} &\sum_{i=1}^{3} \max\left\{\pi - \frac{K^2}{\pi\delta^2(a_i, w, O)}, 0\right\} + \\ &+2\sum_{i=4}^{\infty} \arcsin\left(\tfrac{1}{2}\exp\left\{-\frac{K^2}{\delta^2(a_i, w, O)}\right\}\right) \le \pi \end{aligned} \tag{10.11}$$

where

$$K = 2\sqrt{\pi}\left(h_0 \frac{\nu_2^*}{\nu_1} + 2\,\mathrm{mes}_2\,(D)\right)^{1/2}.$$

Example 10.12 Suppose that $|f| < 1$ everywhere in D. We put

$$h(t) = \begin{cases} 1 & \text{for} \quad t \in [-1, 1]\,, \\ 0 & \text{for} \quad |t| > 1\,. \end{cases}$$

For the auxiliary function

$$w(x) = \int\limits_{-\infty}^{f(x)} h(t)\, dt\,,$$

we have $0 \leq w(x) \leq 2$. Points of quasi-continuity of $f(x)$ are points of quasi-continuity of $w(x)$. Moreover,

$$\delta(a_k, w, O) = \delta(a_k, f, O) \quad \text{for all} \quad k = 1, 2, \ldots .$$

$\square$

Example 10.13 In the case of an unbounded function $f(x)$, we can put

$$h(t) = \frac{1}{1+t^2}.$$

Then

$$w(x) = \int\limits_{-\infty}^{f(x)} \frac{dt}{1+t^2} = \operatorname{arctg} f(x) + \frac{\pi}{2}.$$

Here Theorem 10.10 guarantees that $\operatorname{arctg} f(x)$ is quasi-continuous. It is clear that in the general case, the quasi-continuity of this function does not imply the quasi-continuity of $f(x)$. The function $f(x)$ from Example 10.8 is not quasi-continuous in every boundary point. However, $\operatorname{arctg} f(x)$ is quasi-continuous everywhere on the boundary of the square except its vertices.

$\square$

As a corollary of Theorem 10.10, we have the following statement.

Theorem 10.14 *Let $\mathcal{E} = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 > 1\}$ be an exterior of a unit circle C. Let $D \subset \mathcal{E}$ be a domain and $\Gamma \subset (C \cap \partial D)$ be a boundary arc.*

Then every solution f of the minimal surface equation (2.20) *in D is continuously extendable at an arbitrary point $a \in \Gamma$, except at most, in a countable set.*

Open questions 10.15 1) Find analogs of Theorem 10.14 for solutions of minimal surface type equations. 2) Prove (or refute) Theorem 10.14 with the exchange in its formulation the boundary circle replaced by a concave curve of a sufficient general form.

10.2 The Auxiliary Conformal Map

10.2.1 Definitions and Properties

Let Ω be a graph of a locally Lipschitz function $x_3 = f(x_1, x_2)$, defined over a domain $D \subset \mathbb{R}^2$, and let

$$ds_\Omega^2 = (1 + f'^2_{x_1})\, dx_1^2 + 2 f'_{x_1} f'_{x_2}\, dx_1\, dx_2 + (1 + f'^2_{x_2})\, dx_2^2 \tag{10.16}$$

be the square of its line element on Ω.

Fix arbitrarily a point $a \in D$ where f is differentiable. The function $f \in \operatorname{Lip} D$, and hence, by Rademacher's theorem, it is differentiable a.e. in D.

The quadratic form ds_Ω^2 is positive determined. Thus, an infinitesimal circle with respect to ds_Ω^2 and a center at a is an infinitesimal ellipse with respect to the Euclidean metric. Let $\theta(a)$, $0 \le \theta < \pi$, and $p(a)$, $p \ge 1$, be characteristics of this ellipse (i.e. the angle between its largest axis and the Ox_1-axis, and the ratio of the largest axis to the smaller, respectively).

It is not difficult to see that

$$\theta(a) = \frac{\pi}{2} + \operatorname{arctg} \frac{f'_{x_2}(a)}{f'_{x_1}(a)}, \quad p(a) = \sqrt{1 + |\nabla f(a)|^2}.$$

We consider a quasi-conformal map $u = u(x)$ of the domain D into the $u = (u_1, u_2)$-plane with characteristics which coincide a.e. in D with $\theta(x)$ and $p(x)$. Since $\operatorname{ess\,sup}_{D'} p(x) < \infty$ for every bounded subdomain D', $\overline{D'} \subset D$, then such map exists and defined up to conformal maps in the $u-$plane (see Theorem 3.49).

We assume that the following properties of quasi-conformal maps $u : D \to \mathbb{R}^2$ are known.

(1) A map $u = u(x)$ is differentiable a.e. in D, and at every point of the differentiability it transforms infinitesimal ellipses with characteristics $\theta(x)$, $p(x)$ onto infinitesimal circles (see [17, §**1**]).

(2) A map $u = u(x)$ has the first generalized Sobolev derivatives, square integrable locally in D, i.e. of $W^{1,2}_{\text{loc}}(D)$ (see [17, §**4**]).

(3) Denote by $G = u(D)$ the image of D under $u = u(x)$, and by $x = x(u) = (x_1(u), x_2(u))$ - the inverse map. The characteristic $p(u)$ of the inverse map $x : G \to D$ is locally bounded in the domain G (see [17, §**1**]), and by the property (2) the map $x = x(u)$ belongs to $W^{1,2}_{\text{loc}}(G)$ also.

We put $x_3(u) = f(x(u))$. Since f is locally Lipschitz in D, then by the property (3) the function $x_3(u) \in W^{1,2}_{\text{loc}}(G)$.

From (2) it follows that the vector function

$$\chi(u) = (x_1(u), x_2(u), x_3(u))$$

also belongs to $W^{1,2}_{\mathrm{loc}}(G)$. It is clear that $\chi(u)$ is differentiable a.e.

The vector function $\chi : G \to \Omega$ realizes a bijective map of $G = u(D)$ onto Ω. Let $a \in G$ be a point where χ is differentiable. Denote by $j : F \to D$ the projection of Ω onto D. The map $\chi : G \to \Omega$ is a composition of maps j^{-1} and $x(u)$. Thus $\chi : G \to \Omega$ is conformal a.e. and a.e. on G it has properties (**??**). Variables u_1, u_2 are isothermal coordinates on Ω.

10.2.2 Conformal Types of Surfaces

By the stated above, we say that a surface Ω is of parabolic conformal type if $G = \mathbb{R}^2$ and of hyperbolic conformal type if $G \neq \mathbb{R}^2$.

Fix a function $h \in \mathcal{H}$. Denote by W the graph of the function

$$w(x) = \int\limits_{-\infty}^{f(x)} h(t)\, dt.$$

Theorem 10.17 *Let $D \subset \mathbb{R}^2$ be a simply connected domain bounded by a simple Jordan rectifiable curve. If f is a generalized solution of an equation* (10.4) *in D with structure restrictions* (10.1), (10.2), (10.3), *then for every $h \in \mathcal{H}$, the graph W of $x_3 = w(x)$ is of hyperbolic conformal type.*

Some indications of parabolicity and hyperbolicity of the conformal type of a function graph can be obtained from Theorem 1.36. Below we formulate two such indications.

Theorem 10.18 *Let f be a locally Lipschitz function and let Ω be its graph.*

If Ω is defined over a disc $|x| < R$, $0 < R \leq \infty$, and for a number $0 < r < R$ it is fulfilled

$$\iint\limits_{r<|x|<R} \frac{1 + \langle \nabla f, x^{\perp} \rangle^2}{\sqrt{1 + |\nabla f|^2}} \, \frac{dx_1 dx_2}{|x|^2} < \infty \,, \tag{10.19}$$

then Ω is of hyperbolic conformal type.

If Ω is defined over the plane $\mathbb{R}^2$ and

$$\varliminf_{R\to\infty} \frac{1}{R^2} \iint\limits_{1<|x|<R} \frac{1 + \langle \nabla f, (x/|x|) \rangle^2}{\sqrt{1 + |\nabla f|^2}} \, dx_1 dx_2 = 0 \,, \tag{10.20}$$

then Ω is of parabolic conformal type.

Here by $x^{\perp}$ we denote a unit vector in $\mathbb{R}^2$, orthogonal to the vector x and forming the angle $3\pi/2$ in the direction from $x^{\perp}$ to x.

The **proof** is not difficult. Fix $0 < r < R < \infty$. Let $\Gamma(r,R)$ be the family of all locally rectifiable arcs γ, joining boundary circles and lying in the ring $\{r < |x| < R\}$. Because $\mathrm{mod}_\Omega\,\Gamma(r,R)$ is the conformal invariant, then Ω is of parabolic type if and only if $\mathrm{mod}_\Omega\,\Gamma(r,R) = 0$. By the remark of Section 1.7, we have

$$\left(\int\limits_r^R \frac{d\tau}{\tau \int\limits_0^{2\pi} \frac{1 + \langle \nabla f, (x/|x|)\rangle^2}{\sqrt{1+|\nabla f|^2}}\, d\theta} \right)^{-1} \geq$$

$$\geq \mathrm{mod}_\Omega \Gamma(r,R) \geq \int\limits_0^{2\pi} \frac{d\theta}{\int\limits_r^R \frac{1 + \langle \nabla f, x^{\perp}\rangle^2}{\tau\sqrt{1+|\nabla f|^2}}\, d\tau} \,.$$

By the Cauchy integral inequality, we obtain

$$(R-r)^2 \leq \int\limits_r^R \frac{d\tau}{\tau \int\limits_0^{2\pi} \frac{1 + \langle \nabla f, (x/|x|)\rangle^2}{\sqrt{1+|\nabla f|^2}}\, d\theta} \times$$

$$\times \int\limits_r^R \tau\, d\tau \int\limits_0^{2\pi} \frac{1 + \langle \nabla f, (x/|x|)\rangle^2}{\sqrt{1+|\nabla f|^2}}\, d\theta \,.$$

Thus

$$\frac{(R-r)^2}{\iint\limits_{r<|x|<R} \frac{1 + \langle \nabla f, (x/|x|)\rangle^2}{\sqrt{1+|\nabla f|^2}}\, dx_1 dx_2} \leq \int\limits_r^R \frac{d\tau}{\tau \int\limits_0^{2\pi} \frac{1 + \langle \nabla f, (x/|x|)\rangle^2}{\sqrt{1+|\nabla f|^2}}\, d\theta}$$

that proves (10.20).

For the proof of (10.18), we observe that from Cauchy's inequality it follows

$$\left(\int_0^{2\pi} d\theta\right)^2 \le \int_0^{2\pi} \frac{d\theta}{\tau \int_r^R \frac{1+\langle \nabla f, x^\perp\rangle^2}{\sqrt{1+|\nabla f|^2}}\, d\tau} \int_0^{2\pi} \tau\, d\theta \int_r^R \frac{1+\langle \nabla f, x^\perp\rangle^2}{\sqrt{1+|\nabla f|^2}}\, d\tau$$

and

$$\frac{4\pi^2}{\int_0^{2\pi} d\theta \int_r^R \frac{1+\langle \nabla f, x^\perp\rangle^2}{\tau\sqrt{1+|\nabla f|^2}}\, d\tau} \le \int_0^{2\pi} \frac{\tau\, d\theta}{\int_r^R \frac{1+\langle \nabla f, x^\perp\rangle^2}{\sqrt{1+|\nabla f|^2}}\, d\tau}.$$

Thus we deduce

$$\frac{4\pi^2}{\iint\limits_{r<|x|<R} \frac{1+\langle \nabla f, x^\perp\rangle^2}{\sqrt{1+|\nabla f|^2}} \frac{dx_1 dx_2}{|x|^2}} \le \int_0^{2\pi} \frac{\tau\, d\theta}{\int_r^R \frac{1+\langle \nabla f, x^\perp\rangle^2}{\sqrt{1+|\nabla f|^2}}\, d\tau}.$$

The condition (10.19) implies that $\mathrm{mod}_\Omega \Gamma(r,R) > 0$ and Ω is of parabolic conformal type. □

On indications of conformal type, see also [162], [126], [110], [51], [167] etc.

10.2.3 Some Properties of the Relative Distance

Let $\Omega \subset \mathbb{R}^3$ be a graph of a locally Lipschitz function f defined over a simply connected domain $D \subset \mathbb{R}^2$, and let $O' \in \Omega$ be a fixed point.

For a pair of points $p, q \in \Omega$, let $\rho(p, q; O', \Omega)$ be the relative distance, introduced in Chapter 4. By $\partial\tilde{\Omega}$, as above, we denote the boundary of Ω with respect to the metric ρ, i.e. it will be the set of all sequences $\{p_n\}$ of points in Ω, fundamental with respect to the relative metric ρ and having no accumulation points in Ω.

If Ω is the graph of the function $x_3 \equiv$ const over the domain $D \subset \mathbb{R}^2$, then we identify Ω with D, and the relative boundary $\partial\tilde{\Omega}$ coincides with the Carathéodory boundary of D.

Suppose that D is simply connected and its image $G = u(D)$ is different from the entire plane $\mathbb{R}^2$. Because relations (2.3) are invariant under conformal transforms in the $u = (u_1, u_2)$-plane, then without loss of generality, we can assume that the domain G is bounded.

Consider the above described conformal map

$$\chi(u) = (x_1(u), x_2(u), x_3(u)) : G \to \Omega, \quad u = (u_1, u_2)\,. \tag{10.21}$$

We put $O'' = \chi^{-1}(O')$ and denote by $r(G)$ the Euclidean distance from O'' to ∂G.

The following statement is a special case of Theorem 4.6.

Theorem 10.22 *If the area* $\mathrm{mes}_2\,(\Omega) < \infty$*, then for an arbitrary pair of points* $p, q \in G$*, satisfying the condition*

$$\rho(p, q; O'', G) < \min\{1, \frac{1}{16}\, r^4(G)\}, \tag{10.23}$$

it is fulfilled

$$\rho(\chi(p), \chi(q); O', \Omega) \le K \log^{-1/2} \frac{1}{\rho(p, q; O'', G)}. \tag{10.24}$$

Here

$$K = 2\sqrt{\pi\, \mathrm{mes}_2\,(\Omega)}.$$

For the **proof**, it is sufficient to remark that the surface Ω, given by the graph of a locally Lipschitz function f, is locally bi-Lipschitz. It is obvious, since for every subdomain $D' \subset\subset D$, the relation (10.16) implies

$$|dx|^2 \le ds_\Omega^2 = (1 + f'^2_{x_1})\, dx_1^2 + 2 f'_{x_1}\, f'_{x_2}\, dx_1\, dx_2 + (1 + f'^2_{x_2})\, dx_2^2 \le C(f, D')\, |dx|^2$$

where $C(f, D')$ is a constant. □

The estimate (10.24) implies that every fundamental (with respect to the relative distance $\rho(p, q; O'', G)$) sequence $\{a_n\} \subset G$ turns to a fundamental (with respect to the metric $\rho(\chi(p), \chi(q); O', \Omega)$) sequence $\{\chi(a_n)\} \subset \Omega$. Thus we obtain the following statement.

Corollary 10.25 *Under conditions described in Theorem* 10.14*, a conformal mapping* (10.21) *is continuously extended up to a continuous mapping of the relative boundary* $\partial\tilde{G}$ *onto the relative boundary* $\partial\tilde{\Omega}$*.*

It should be noted that the following important property of the projection $j : F \to D$.

Lemma 10.26 *Let Ω be the graph of a locally Lipschitz function, defined over a simply connected domain $D \subset \mathbb{R}^2$, and $j(O') = O$.*

Then for every pair of points $p, q \in \Omega$, it fulfills

$$\rho(j(p),\, j(q); O, D) \leq \rho(p,\, q; O', \Omega)\,. \tag{10.27}$$

Proof. It is sufficient to remark that the projection $j : F \to D$ does not increase lengths of curves. □

Let $x(u) = (x_1(u), x_2(u)) : G \to D$ be a map realized by components $x_1(u)$, $x_2(u)$ of the vector function (10.21).

Corollary 10.28 *If $\mathrm{mes}_2\,(\Omega) < \infty$, then for every pair of points $p, q \in G$ with the property* (10.23)*, it fulfills*

$$\rho(x(p),\, x(q); O, D) \leq K \log^{-1/2} \frac{1}{\rho(p,\, q; O'', G)}$$

where K is the constant of Theorem 10.22*.*

The **proof** follows from Theorem 10.22 and Lemma 10.26. □

10.3 Solution Jumps on a Boundary

We prove Theorem 10.10.

At first we observe that the statement is trivial if $f \equiv \mathrm{const}$. Thus we can assume that $f \not\equiv \mathrm{const}$.

10.3.1 The Estimate of an Area Graph

Let f be a locally Lipschitz solution of the equation (10.4) with structure conditions (10.1)-(10.3). We assume that the domain D is simply connected and bounded with a simple Jordan rectifiable curve.

Lemma 10.29 *Under described suppositions, the following inequality holds*

$$\mathrm{mes}_2\,(W) \leq h_0 \nu + 2\,\mathrm{mes}_2\,(D). \tag{10.30}$$

Proof. Fix a sequence of domains D_k, $k = 1, 2, \dots$, such that

$$\overline{D}_k \subset D_{k+1}, \quad \cup_{k=1}^{\infty} D_k = D.$$

The function $w(x)$ belongs to the class $\mathrm{Lip}\,(\overline{D_k})$, $k = 1, 2, \dots$. Choosing $\varphi = w(x)$ in (10.5), we have

$$\iint\limits_{D_k} \sum_{i=1}^{2} f'_{x_i} A_i(x, \nabla f)\, h(f(x))\, dx_1\, dx_2 = \int\limits_{\partial D_k} w(x) \sum_{i=1}^{2} A_i(x, \nabla f)\, n_i\, |dx|$$

$$\le \int\limits_{\partial D_k} w(x)\, |A(x, \nabla f)|\, |dx|.$$

Since $0 < w(x) \le h_0 < \infty$, then

$$\iint\limits_{D_k} \sum_{i=1}^{2} f'_{x_i} A_i(x, \nabla f)\, h(f(x))\, dx_1\, dx_2 \le h_0 \int\limits_{\partial D_k} |A(x, \nabla f)|\, |dx|.$$

Passing to the limit as $k \to \infty$ and using the structure conditions (10.1)-(10.3), we find

$$\nu_1 \iint\limits_{D} \frac{|\nabla f|^2}{\sqrt{1 + |\nabla f|^2}}\, h(f(x))\, dx_1\, dx_2 \le h_0\, \nu_2^*. \tag{10.31}$$

From (10.31) we deduce

$$\iint\limits_{D} \sqrt{1 + |\nabla f|^2} h(f(x))\, dx_1\, dx_2 \le h_0 \frac{\nu_2^*}{\nu_1} + \iint\limits_{D} \frac{h(f(x))}{\sqrt{1 + |\nabla f|^2}}\, dx_1\, dx_2.$$

From here, remembering that $0 \le h(f(x)) \le 1$, we arrive to the estimate

$$\iint\limits_{D} |\nabla w(x)|\, dx_1\, dx_2 = \iint\limits_{D} |\nabla f(x)|\, h(f(x))\, dx_1\, dx_2 \le h_0 \frac{\nu_2^*}{\nu_1} + \mathrm{area}\,(D).$$

Because

$$\sqrt{1 + |\nabla w(x)|^2} < 1 + |\nabla w(x)|,$$

we find finally

$$\iint\limits_{D} \sqrt{1 + |\nabla w|^2}\, dx_1\, dx_2 \le h_0 \frac{\nu_2^*}{\nu_1} + 2\, \mathrm{area}\,(D)$$

that is equivalent to (10.30). □

10.3.2 Monotonicity of Solutions

The square of the length element on the surface W is given by the formula

$$ds_W^2 = \sum_{i,j=1}^{2} g_{ij}(x)\, dx_i\, dx_j, \quad g_{ij}(x) = \delta_{ij} + w'_{x_i}\, w'_{x_j}\, dx_i\, dx_j \quad (i, j = 1, 2)$$

where δ_{ij} $(i, j = 1, 2)$ is the Kronecker symbol.

Let $(g^{ij}(x)) = (g_{ij})^{-1}(x)$ be the matrix inverse to (g_{ij}). Simple calculations imply

$$g^{ij} = \delta_{ij} - \frac{w'_{x_i} w'_{x_j}}{1 + |\nabla w|^2}.$$

For an arbitrary $\xi \in \mathbb{R}^2$, we put

$$|\xi|_W^2 = \sum_{i,j=1}^{2} g^{ij}(x)\, \xi_i \xi_j.$$

It is easy to check that

$$|\nabla w|_W^2 = \frac{1 + w'^2_{x_2}}{1 + |\nabla w|^2}\, w'^2_{x_1} - 2\frac{w'_{x_1}\, w'_{x_2}}{1 + |\nabla w|^2}\, w'_{x_1}\, w'_{x_2} +$$

$$+ \frac{1 + w'^2_{x_1}}{1 + |\nabla w|^2}\, w'^2_{x_2} = \frac{|\nabla w|^2}{1 + |\nabla w|^2}\,.$$

Since $\nabla w(x) = h(f(x))\, \nabla f(x)$, the relation (10.31) implies

$$\iint\limits_D \frac{|\nabla w|^2}{\sqrt{h^2(f) + |\nabla w|^2}}\, dx_1\, dx_2 =$$

$$\iint\limits_D \frac{|\nabla f|^2}{\sqrt{1 + |\nabla f|^2}}\, h(f(x))\, dx_1\, dx_2 \le h_0\, \frac{\nu_2^*}{\nu_1}.$$

From here, taking into account that $h(f(x)) \le 1$, we arrive to the inequality

$$\iint\limits_D \frac{|\nabla w|^2}{\sqrt{1 + |\nabla w|^2}}\, dx_1\, dx_2 \le h_0\, \frac{\nu_2^*}{\nu_1},$$

or

$$\iint\limits_D |\nabla w|_W^2 \sqrt{1 + |\nabla w|^2}\, dx_1\, dx_2 \le h_0\, \nu. \tag{10.32}$$

Lemma 10.33 *Every generalized solution f of the equation* (10.4) *with structure conditions* (10.1), (10.2), (10.3) *is monotone in the Lebes- gue sense.*

Proof. Indeed, for example, suppose that $a \in D$ is a point of the strong local maximum. For sufficiently small $\epsilon > 0$, the set $\{x \in D : f(x) > f(a) - \epsilon\}$ contains a precompact component $U \subset D$, $a \in U$.

We have $f(x) - f(a) + \epsilon = 0$ on the boundary of U. For a.e. $\epsilon > 0$, curves ∂U are rectifiable. Choosing in (10.5) the function $\phi = f(x) - f(a) + \epsilon$, for a.e. $\epsilon > 0$ we can write

$$\iint\limits_U \sum_{i=1}^2 f'_{x_i}\, A_i(x, \nabla f)\, dx_1 dx_2 =$$

$$\int\limits_{\partial U} (f(x) - f(a) + \epsilon) \sum_{i=1}^2 A_i(x, \nabla f)\, n_i\, |dx| = 0.$$

Thus from (10.1), it follows

$$\iint\limits_U \frac{|\nabla f|^2}{\sqrt{1 + |\nabla f|^2}}\, dx_1 dx_2 = 0,$$

i.e. $\nabla f \equiv 0$ and $f \equiv \text{const}$ in U. We have the contradiction with the assumption that a is the point of the strong local maximum. $\square$

10.3.3 Proof of Theorem 10.17

As above, with an auxiliary quasi-conformal map $u = u(x) : D \to \mathbb{R}^2$, we introduce isothermal coordinates (u_1, u_2) on the surface W.

Consider the mapping $x = x(u) = (x_1(u), x_2(u))$, inverse to $u = u(x)$, and the function $x_3(u) = w(x(u))$. The vector function

$$\chi(u) = (x_1(u), x_2(u), x_3(u))$$

realizes the one-to-one conformal mapping of $G = u(D)$ onto W.

Complete the proof of the theorem. At first we remark that by Lemma 10.33 the function $w(x)$ is monotone in D, and since the map $x(u) : D \to G$ is homeomorphic, then $w^*(u) = w(x(u))$ is also monotone in G. Because $x(u)$ is quasi-conformal and f is locally Lipschitz, then $w^* \in W^{1,2}_{\text{loc}}(G)$.

Suppose that the function w^* is defined over the whole of plane $\mathbb{R}^2$. Fix $R > 0$ and $t > 1$. Let $B(\tau)$ and $S(\tau)$ be the disc and the circle with the center at $u = 0$ and the radius τ, respectively.

As the proved above inequality (4.14), we prove that

$$\inf_{R\le\tau\le tR} \operatorname{osc}(S(\tau),\, w^*) \le \left(\frac{2\pi\, I(R)}{\log t}\right)^{1/2} \tag{10.34}$$

where

$$I(R) = \iint\limits_{|u|<tR} |\nabla w^*|^2 du_1\, du_2.$$

The function w^* is monotone, and hence, for every $t > 1$, it fulfills

$$\operatorname{osc}(B(R),\, w^*) \le \operatorname{osc}(S(R),\, w^*) \le \inf_{R\le\tau\le tR} \operatorname{osc}(S(\tau),\, w^*).$$

Consequently,

$$\operatorname{osc}(B(R),\, w^*) \le \left(\frac{2\pi\, I(\infty)}{\log t}\right)^{1/2}.$$

Taking into account (10.32), we find

$$I(\infty) = \iint\limits_{\mathbb{R}^2} |\nabla w^*|^2\, du_1\, du_2 = \iint\limits_{D} |\nabla w|_W^2 \sqrt{1+|\nabla w|^2}\, dx_1\, dx_2 \le h_0\, \nu,$$

and hence,

$$\operatorname{osc}(B(R),\, w^*) \le \left(\frac{2\pi\, h_0\, \nu}{\log t}\right)^{1/2}.$$

Setting now $t \to \infty$, we conclude that $w^* \equiv \text{const}$ on $B(R)$. But $R > 0$ is arbitrary, and hence, $w^* \equiv \text{const}$ in the whole of plane $\mathbb{R}^2$. From here, it follows that $w \equiv \text{const}$ in D.

However, if $w \equiv \text{const}$ in D, then the map $u = u(x) : D \to \mathbb{R}^2$ is conformal with respect to the Euclidean metric, and the image of D can not be the whole of plane $\mathbb{R}^2$. We have a contradiction. □

10.3.4 Points of Quasi-continuity

Because the simply connected domain $G \ne \mathbb{R}^2$ and the mapping $u(x) : D \to G$ is defined up to conformal mappings in the $u-$plane, we are right to assume that G is a unit disc with the center at O''.

We will show that *all points of ∂G are points of the quasi-continuity* of w^*.

Fix a point $a \in \partial G$. Let $S(a,\tau)$ be a component of the set

$$\{u \in G : |u - a| = \tau\}$$

separating the point $a \in \partial G$ from the origin O'', $0 < \tau < 1$.

As in (10.34), we prove that for every $R \in (0,1)$ and every $t \in (0,1)$, there exists $\tau_0 \in (tR, R)$ such that

$$\inf_{tR<\tau<R} \operatorname{osc}\left(S(a,\tau),\, w^*\right) \leq \left(\frac{2\pi\, I}{\log \frac{1}{t}}\right)^{1/2}$$

where

$$I = \iint\limits_{G} |\nabla w^*|^2 \, du_1 \, du_2 \, .$$

Choosing now $t_n \to 0$, we find a sequence $\tau_n \to 0$ along which

$$\operatorname{osc}(S(a,\tau_n),\, w^*) \to 0\, . \tag{10.35}$$

This means that a is the quasi-continuity point of w^* at $a \in \partial G$.

By Lemma 10.29 the integral

$$\iint\limits_{D} \sqrt{1 + |\nabla w|^2}\, dx_1 \, dx_2 < \infty.$$

Using Theorem 10.14 and Lemma 10.26, we conclude that the mapping

$$x(u) = j \circ \chi(u) : G \to D$$

is continuous up to the boundary of the unit disc ∂G.

Denote by $\tilde{x}(u) : \overline{G} \to \overline{D},\ \tilde{x}(u)|_G = x(u)$, the map between closed domains obtained with the continuous extension of $x(u) : G \to D$ to ∂G.

Moreover, by Corollary 10.28 for arbitrary points

$$p,\, q \in \overline{G} \quad \text{with} \quad \rho(p,\, q; O'', G) < 1/16\, ,$$

we have

$$\rho(\tilde{x}(p),\, \tilde{x}(q);\, O, D) \leq K_1 \log^{-1/2} \frac{1}{\rho(p,\, q;\, O'', G)}$$

where

$$K_1 = 2\sqrt{\pi\, \mathrm{mes}_2\, (W)}$$

is the constant.

By Lemma 10.29

$$K_1 \leq 2\sqrt{\pi}\, (\nu\, h_0 + 2\, \mathrm{area}\, (D))^{1/2} = K\, .$$

In particular, if points p and q lie on the boundary circle and $|p-q| < 1/16$, then $\rho(p,\, q; O'', G) = |p-q|$. Thus the found estimate takes the form

$$\rho(\tilde{x}(p),\, \tilde{x}(q); O, D) \leq K \log^{-1/2} \frac{1}{|p-q|} \qquad |p-q| < \frac{1}{16}\, . \tag{10.36}$$

Since the mapping $\tilde{x}(u) : \overline{G} \to \overline{D}$ is continuous, then the preimage $\tilde{x}^{-1}(a)$ of a point $a \in \partial D$ is a closed set on the unit circle ∂G. Moreover, for every pair of points $a \neq b$ on ∂D, it is fulfilled

$$\tilde{x}^{-1}(a) \cap \tilde{x}^{-1}(b) = \emptyset\,.$$

The set $\tilde{x}^{-1}(a)$ is connected. Indeed, because ∂D is the simple Jordan curve, for sufficiently small $\epsilon > 0$ the set $U_\epsilon = \{x \in D : |x - a| < \epsilon\}$ is connected. The mapping $x(u) : G \to D$ is homeomorphic, and hence, preimages $x^{-1}(U_\epsilon)$ are connected for all $\epsilon \in (0, \epsilon_0)$ where $\epsilon_0 > 0$ is a sufficiently small number. Thus the set

$$\tilde{x}^{-1}(a) = \cap_{\epsilon > 0} \overline{U}_\epsilon$$

is also connected.

So, for an arbitrary point $a \in \partial D$, the set $\tilde{x}^{-1}(a)$ can be either an isolated point or a closed connected arc.

Since the amount of non-overlapping arcs on a circle is no more than countable, then the inverse mapping $\tilde{x}^{-1}(x)$ is defined everywhere on ∂D, except, at most, in a countable set $E \subset \partial G$.

We show that *every point $a \in \partial D \setminus E$ is the quasi-continuity point* of the function $w(x)$.

Let $b = \tilde{x}^{-1}(a)$ be the preimage of the point $a \in \partial D \setminus E$. By (10.35) there exists a sequence of subdomains $\{G_n\}$, $\partial' G_n = S(b, \tau_n)$ with properties:

every arc $\partial' G_n$ separates b from the origin O;

length $(\partial' G_n) \to 0$ and osc $(\partial' G_n, w^*) \to 0$ as $n \to \infty$;

finally, $\cap_{n=1}^{\infty} [G_n] = \emptyset$.

We put $D_n = x(G_n)$, $n = 1, 2, \ldots$. Because the map $x(u) : G \to D$ is homeomorphic, every arc $\partial D_n = x(S(b, \tau_n))$ separates $a \in \partial D$ from O.

The map $\tilde{x}(u) : \overline{G} \to \overline{D}$ is continuous, and hence, length $\partial' D_n \to 0$ and $\cap_{n=1}^{\infty} [D_n] = \emptyset$.

Finally, osc $(\partial' D_n, w) = \mathrm{osc}(S(b, \tau_n), w^*)$, and hence, osc $(\partial' D_n, w) \to 0$ as $n \to \infty$.

Thus the point $a \in \partial D \setminus E$ has all properties, necessary to define the points of quasi-continuity of $w(x)$.

10.3.5 Behavior of Solutions at Jump Points

Fix a jump point $a \in E$. Its preimage $\tilde{x}^{-1}(a)$ is a subarc β of the circle $|u| = 1$. We estimate its length $l(\beta)$. Let $\xi \in \beta$ be the middle of the arc. It is easy now to calculate

$$r = 2\sin\frac{l(\beta)}{4}\,,$$

which would be the distance from ξ to arc ends.

Suppose that $r < 1$, i.e. $l(\beta) < \frac{2}{3}\pi$. Consider the family of circles $\{S(\xi,\tau)\}$ with the center at ξ and radii τ, $r < \tau < 1$. We put $C_\tau = G \cap S(\xi,\tau)$.

By (4.12) we have

$$\int_r^1 \frac{l^2(\chi(C_\tau))}{\tau}\,d\tau \le 2\pi \iint_{\Delta_{r,1}} \sum_{i=1}^{3} |\nabla x_i(u)|^2\,du_1\,du_2.$$

Remark now that for every $\tau \in [r,1]$, it is fulfilled

$$l(\chi(C_\tau)) \ge \max\{\mathrm{osc}\,(x(C_\tau),w), \mathrm{length}\,(x(C_\tau))\} \ge \delta(a,w,O),$$

and hence,

$$\delta^2(a,\,w,\,O) \le 4\pi\,\log^{-1}\frac{1}{r}\,\mathrm{mes}_2\,(W).$$

Thus, taking into account the estimate (10.30), we arrive to the inequality

$$\delta^2(a,w,O) \le \frac{K^2}{\log 1/r}$$

where K is the constant of Theorem 10.10.

From here,

$$r = 2\sin\frac{l(\beta)}{4} \ge \exp\{-\frac{K^2}{\delta^2(a,w,O)}\},$$

and we have the following estimate of $l(\beta)$ for 'small' $\delta(a,w,O)$:

$$l(\beta) \ge 4\arcsin\left(\frac{1}{2}\exp\{-\frac{K^2}{\delta^2(a,w,O)}\}\right) \quad \text{for} \quad l(\beta) < \frac{2}{3}\pi. \tag{10.37}$$

Find the estimate of $l(\beta)$ for 'big' $\delta(a,w,O)$.

Let $a \in \partial D \cap E$ be a jump point in which $\frac{2}{3}\pi \le l(\beta) \le 2\pi$. Without loss of generality, we can assume that the arc $\beta = \tilde{x}^{-1}(a)$ is described by relations

$$\beta = \{u = (u_1,u_2) : u_1^2 + u_2^2 = 1,\, -\frac{l(\beta)}{2} \le \mathrm{arctg}\,\frac{u_2}{u_1} \le \frac{l(\beta)}{2}\}.$$

Fix the segment

$$\gamma = \{u = (u_1,u_2) : 0 \le u_1 \le 1,\, u_2 = 0\}$$

and denote by p_1 and p_2 the boundary points of the unit disc with the cut γ, lying in the intersection of the circle $|u| = 1$ and the upper and lower edges of γ respectively.

Let $T : G \to V$ be the conformal mapping of the disc $|u| < 1$ with the cut γ onto the half-disc $V = \{v = (v_1, v_2) : v_1^2 + v_2^2 < 1, \, v_2 > 0\}$ for which $T(p_1) = (1, 0)$, $T(p_2) = (-1, 0)$ and $T(0, 0) = (0, 0)$.

Under the map T, the arc $\beta \subset \partial G$ is corresponding to

$$\eta = \left\{ v = (v_1, v_2) : \frac{l(\beta)}{4} \le \operatorname{arctg} \frac{v_2}{v_1} \le \pi - \frac{l(\beta)}{4} \right\} .$$

For every $0 < k < \cos \frac{l(\beta)}{4}$, the rectilinear segment

$$\zeta(k) = \{v = (v_1, v_2) : -\sqrt{1 - k^2} < v_1 < \sqrt{1 - k^2}, \; v_2 = k\}$$

separates in the domain V the arc η from the point $(0, 0)$. Its image $\zeta^*(k) = \tilde{x} \circ T^{-1}(\zeta(k))$ is an arc, separating points a and O in D. Thus for every $k \in (0, \cos \frac{l(\beta)}{4})$ it is fulfilled

$$l^2(\zeta^*(k)) \le \left(\int\limits_{\zeta(k)} \left| \frac{\partial \chi^*}{\partial v_1} \right| dv_1 \right)^2 \le 2\sqrt{1 - k^2} \int\limits_{\zeta(k)} \left| \frac{\partial \chi^*}{\partial v_1} \right|^2 dv_1$$

where $\chi^* = \chi \circ T^{-1}$.

Further,

$$\int\limits_0^{\cos l(\beta)/4} \frac{l^2(\zeta^*(k))}{\sqrt{1 - k^2}} \, dk \le 2 \iint\limits_V \left| \frac{\partial \chi^*}{\partial v_1} \right|^2 dv_1 \, dv_2 \le 2 \operatorname{mes}_2 (W).$$

Taking into account that

$$l(\zeta^*(k)) \ge \delta(a, \, w, \, O) \quad \text{for all} \quad k \in (0, \cos \frac{l(\beta)}{4}) .$$

From here we obtain

$$\delta^2(a, w, O) \int\limits_0^{\cos l(\beta)/4} \frac{1}{\sqrt{1 - k^2}} \, dk \le 2 \operatorname{mes}_2 (W) \, ,$$

or

$$\delta^2(a, w, O) \left(\frac{\pi}{2} - \frac{l(\beta)}{4} \right) \le \frac{K^2}{2\pi}$$

where K is the constant of Theorem 10.10.

Thus we arrive to the estimate

$$l(\beta) \geq \max\left\{2\pi - \frac{2K^2}{\pi\,\delta^2(a,w,O)},\ 0\right\}. \tag{10.38}$$

10.3.6 Estimate of a Summary Jump

Let a_k be jump points of the function $w(x)$ and let β_k be corresponding closed arcs on $|u| = 1$, $k = 1, 2, \ldots$. We are right to assume that

$$l(\beta_1) \geq l(\beta_2) \geq \ldots \geq l(\beta_k) \geq \ldots.$$

Since $\beta_i \cap \beta_j = \emptyset$ for $i \neq j$, then

$$\sum_{k=1}^{\infty} l(\beta_k) \leq 2\pi.$$

Beginning at least with $k = 4$, it is fulfilled $l(\beta_k) < \frac{2}{3}\pi$, and we can use the estimate (10.37). We have

$$4\sum_{k=4}^{\infty} \arcsin\left(\frac{1}{2}\exp\{-\frac{K^2}{\delta^2(a,w,O)}\}\right) \leq \sum_{k=4}^{\infty} l(\beta_k). \tag{10.39}$$

For every point a_1, a_2, a_3, we use the estimate (10.38). Then

$$\sum_{k=1}^{3} \max\{2\pi - \frac{2K^2}{\pi\delta^2(a,w,O)},\ 0\} \leq \sum_{k=1}^{3} l(\beta_k). \tag{10.40}$$

Combining (10.39) and (10.40), we arrive to the inequality

$$\sum_{k=1}^{3} \max\{2\pi - \frac{2K^2}{\pi\,\delta^2(a,w,O)},\ 0\} + 4\sum_{k=4}^{\infty} \arcsin\left(\frac{1}{2}\exp\{-\frac{K^2}{\delta^2(a,w,O)}\}\right)$$

$$\leq \sum_{k=1}^{3} l(\beta_k) + \sum_{k=4}^{\infty} l(\beta_k) \leq 2\pi.$$

Thus Theorem 10.10 is proved completely. □

10.4 The Fatou Type Theorem

Prove Theorem 10.7. Let $f : D \to \mathbb{R}$ be a function, monotone in the Lebesgue sense, and let $a \in \partial D$ be a point. We call that f *has the Lindelöf property at* a if the existence finite limits of f along ways Γ_1 and Γ_2, leading to this point, implies the existence of a limit of f along every way, leading to a and lying between Γ_1 and Γ_2.

The proof of Theorem 10.7 is based on the following statement.

Lemma 10.41 *If a function is monotone in the Lebesgue sense and quasi-continuous at a boundary point, then this function has the Lindelöf property at this point.*

For the **proof** we fix a point of quasi-continuity $a \in \partial D$ and a sequence of subdomains $\{D_k\}_{k=1}^{\infty}$ of D with properties (α), (β), (γ) of the definition 10.9.

Let Γ_1 and Γ_2 be arbitrary ways, leading to a, along which f has limits α_1 and α_2 respectively. Denote by U_k the subdomain of D_k lying between Γ_1, Γ_2, ∂D_k and ∂D_{k+1}. Without loss of generality, we can assume that such subdomain is defined for every $k = 1, 2, \ldots$.

For $k = 1, 2, \ldots$, we put

$$\gamma_{1k} = \partial U_k \cap \Gamma_1, \quad \gamma_{2k} = \partial U_k \cap \Gamma_2, \quad \gamma_{3k} = \partial U_k \cap \partial D_k, \quad \gamma_{4k} = \partial U_k \cap \partial D_{k+1}\,.$$

The function f has finite limits along Γ_1 and Γ_2. Therefore,

$$\operatorname{osc}(\gamma_{ik}, f) \to 0 \quad \text{as} \quad k \to \infty \quad \text{for} \quad i = 1, 2.$$

Moreover, by (γ) of the definition 10.9, we have

$$\operatorname{osc}(\gamma_{ik}, f) \to 0 \quad \text{as} \quad k \to \infty \quad \text{for} \quad i = 3, 4.$$

Thus, taking into account that

$$\partial U_k = \cup_{i=1}^{4} \gamma_{ik},$$

we can conclude

$$\operatorname{osc}(\partial U_k, f) \to 0 \quad \text{as} \quad k \to \infty.$$

However, f is monotone in the Lebesgue sense, and therefore,

$$\operatorname{osc}(U_k, f) \to 0 \quad \text{as} \quad k \to \infty.$$

This means that f has a limit along the sequence $\{U_k\}$ and, in particular, $\alpha_1 = \alpha_2$. Thus f has the Lindelöf property at a. □

Proof of Theorem 10.7. Let f be a solution of (10.4) with structure restrictions (10.1), (10.2), and (10.3) in D. We put $\Phi(x) = \operatorname{arctg} f(x)$.

Choose $h(t)$, as in Example 10.13. By (10.30), we have

$$\iint\limits_D |\nabla\Phi(x)|\, dx_1\, dx_2 < \infty.$$

Denote by D_c the intersection of D and the line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = c\}$, and by Γ_c - the intersection of ∂D and the same line. Using Fubini theorem, we conclude that for a.e. c, belonging to the projection of D onto the axis $0x_1$, the following relation holds

$$\int\limits_{D_c} |\nabla\Phi(c, x_2)|\, dx_2 < \infty.$$

From here for every pair of points (c, x_2'), (c, x_2'') which belong to one and the same connected component of D_c, we obtain

$$|\Phi(c, x_2'') - \Phi(c, x_2')| \le \int\limits_{x_2'}^{x_2''} |\nabla\Phi(c, x_2)|\ dx_2 \quad (x_2' < x_2'').$$

Thus, $\Phi(c, x_2)$ is uniformly continuous on every component of the set D_c, and hence, at every point $a \in \Gamma_c$, it has a limit along D_c.

We turn the coordinate system in the $(x_1, x_2)-$plane in the angle $\alpha \in F$ where F is a countable set dense everywhere on $(0, 2\pi)$, and reason every time as above. Thus we can touch a.e. point $a \in \partial D$ with a corner of any angle, close to π and such that Φ has limits along every side of this corner.

Suppose that $a \in \partial D$ be a point of the described form. By Theorem 10.10 the function $\Phi = \operatorname{arctg} f$ is quasi-continuous everywhere on ∂D except, possibly, in a countable set, and without loss of generality, we are right to assume that a is a point of the quasi-continuity of f. By Lemma 10.33 the function f (and, hence, $\Phi = \operatorname{arctg} f$) is monotone in the Lebesgue sense. Thus by Lemma 10.41, this function has Lindelöf's property at a.

Choose arbitrarily a corner along sides of which Φ has limits. These limits equal a number α, $|\alpha| \le \frac{\pi}{2}$, and the constant α is also the limit of Φ along the corner domain. But magnitude of this corner can be chosen arbitrarily close to π, and hence, Φ has the angular boundary value α.

From here, $f = \operatorname{tg}\Phi$ has the finite or infinite angular boundary value $\operatorname{tg}\alpha$ at this point. $\square$

10.5 Continuity and Quasi-continuity

Below we prove Theorem 10.22. We will need a special case of Finn's lemma [38]. In the form, necessary below, this statement is contained in [141, Lemma **10.2**].

Lemma 10.42 *Let Δ be a subdomain, lying in a ring $1 < |x| < b$, and let γ be a set of boundary points of Δ which are not lying on the unit circle. Let f be a solution of the minimal surface equation* (2.20) *in Δ such that, for all points $x \in \gamma$*

$$m \leq f(x) \leq M$$

where m and M are constants.

Then, everywhere in Δ, the following inequality holds

$$m - \operatorname{arccosh} b \leq f(x) \leq M + \operatorname{arccosh} b. \tag{10.43}$$

Proof of Theorem 10.14. The function f can not be identically to $\pm\infty$ on an arc $\Gamma_1 \subset \Gamma$ (see [141, §**10**]). Thus there exists an everywhere dense set on Γ, such that in its every point, it is possible to touch outwards D with an arc along which f is bounded. Using Lemma 10.42, from here we conclude that in arbitrary strongly inner subarc $\Gamma_1 \subset \Gamma$, limiting values of f are bounded. Thus it is sufficient to prove the statement in case if D is simply connected and bounded by a simple Jordan rectifiable curve and f is bounded in D.

By Theorem 10.10 the function f is quasi-continuous everywhere on ∂D, except, possibly, in a countable set.

Fix a point $a \in \partial D$ where f is quasi-continuous, and an inner point $O \in D$. There exists a sequence $\{\gamma_k\}$, $\gamma_k \subset D$, $k = 1, 2, \ldots$, of arcs with ends on Γ separating the point a from O and such that

$$\lim_{k\to\infty} \operatorname{osc}(\gamma_k,\, f) = 0. \tag{10.44}$$

Denote by D_k the subset of D separating γ_k from the point O. The inequality (10.43) implies that

$$\operatorname{osc}(D_k, f) \leq \operatorname{osc}(\gamma_k, f) + 2\operatorname{arccosh}(1 + \operatorname{length}(\gamma_k)) \quad (k = 1, 2, \ldots).$$

Thus by (10.44) we obtain

$$\lim_{k\to\infty} \operatorname{osc}(D_k, f) = 0\,,$$

and f is continuously extended to the point $a \in \Gamma$. Theorem is proved. □

Open questions 10.45 1) Find a direct (not using auxiliary conformal mappings of a surface to the plane of isothermal coordinates) proof of Theorem 10.10. 2) Study behavior of generalized solutions near points of quasi-continuity and near jump points. In what cases, do limits of solutions along non-tangent ways exist? Along tangent ways? 3) Find multi-dimensional versions of Fatou's type theorem for minimal graphs.

Index

References

[1] D.R. Adams and L.I. Hedberg, *Function Spaces and Potential Theory*, Springer-Verlag, Berlin-Heidelberg-New York etc., 1996.

[2] L.V. Ahlfors, *Untersuchungen zur Theorie der konformen Abbildungen und der ganzen Funktionen*, Acta. Soc. Sci. Fenn. N. S. A I, **1**, n. 9, 1930, 1-40.

[3] L.V. Ahlfors, *Lectures on quasiconformal mappings*, D. Van Nostrand Comp. Inc., Princeton, 1966.

[4] L.V. Ahlfors, *Conformal invariants*, McGraw-Hill, New York, 1973.

[5] R.S. Akopyan, *Stabilization conditions of a minimal surface over half-strip*, in "Scien. schools of Volgograd State univ. Geometric analysis and its applications", izd-vo VolGU, Volgograd, 1999, 105-110 (in Russian).

[6] G. Alessandrini and V. Nessi, *Univalent σ-harmonic mappings*, Arch. Ration. Mech. and Anal., v. 158, 2001, 155-171.

[7] G.S. Asanov, *Finslerlike Geometry*, izd-vo MGU, Moscow, 2004 (in Russian).

[8] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math., **173**, 1994, 37—60.

[9] K. Astala and V. Nesi, *Composites and quasiconformal mappings: new optimal boundes*, Preprint 223, Departm. of Math., University of Jyväskylä, 2000, 26 pp.

[10] F.G. Avkhadiev, *Conformal maps and boundary problems*, Kazan' fund "Mathematics" , 1996, 216 pp.

[11] C. Bandle, *Isoperimetric inequalities and applications*, Pitman Adv. Pub. Program: Boston—London—Melbourne, 1980.

[12] E.F. Beckenbach, R. Bellman, *Inequalities*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961.

[13] P.P. Belinskii, *On distortion under quasiconformal maps*, Dokl. AN SSSR, v. 91, n. 5, 1953, 997-998 (in Russian).

[14] P.P. Belinskii, *On area measure under quasiconformal mappings*, Dokl. AN SSSR, v. 121, n. 1, 1958, 16-17 (in Russian).

[15] P.P. Belinskii, *Solution of extremal problems of the quasiconformal maps theory by the variational method*, Sib. math. j., v. 1, n. 3, 1960, 303-330 (in Russian).

[16] P.P. Belinskii, *On the closeness order of a space quasiconformal map to the conformal map*, Sib. math. j., v. 14, n. 3, 1973, 475-483 (in Russian).

[17] P.P. Belinskii, *General properties of quasiconformal maps*, Nauka, Sib. otd., Novosibirsk, 1974 (in Russian).

[18] V.I. Belyi, V.M. Miklyukov, *Some properties of conformal and quasiconformal maps and direct theorems of the constructive function theory*, Izv. AN SSSR, Ser. math., v. 38, n. 6, 1974, 1343-1361 (in Russian).

[19] S.N. Bernstein, *On a geometrical theorem and its applications to partial differential equations*, Sobr. soch., v. III, Izd-vo AN SSSR, 1960, 251-258 (in Russian).

[20] L. Bers, *Mathematical aspects of subsinic and transonic gas dinamics*, New York · John Wiley & Sons, inc. London · Chapman & hall, Limited, 1958.

[21] L. Bers, *Uniformization by Beltrami equation*, Comm. Pure Appl. Math., v. 14, 1961, 215-228.

[22] A.A. Borisenko, *The inner and exterior geometry of multi dimensional submanifolds*, Izd-vo Ekzamen, Moscow, 2003 (in Russian).

[23] C. Caratheodory, *Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten,* Math. Ann., v. 72, 1912, 107-144.

[24] C. Caratheodory, *Über die Begrenzung einfach zusammenhangender Gebiete*, Math. Ann., v. 73, 1913, 323-370.

[25] C. Caratheodory, *Conformal Representation*, Cambridge at the University Press, 1932.

[26] T. Carleman, *Sur une inégalité differentielle dans la théorie des functions analytiques*, C.R.Acad. Sci. Paris, v. 196, 1933, 995-997.

[27] S. Chandrasekhar, *The mathematical theory of black holes*, Clarendon Press Oxford Oxford University Press New York, 1983.

[28] E.F. Collingwood, A.J. Lohwater, *The theory of cluster sets*, Cambrige Univ. Press, Cambrige, 1966.

[29] R. Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces*, Interscience, New York, 1950.

[30] G. David and P. Mattila, *Removable sets for Lipschitz harmonic functions in the plane*, Revista Matem. Iberoamericana, v. 16, n. 1, 2000, 137-215.

[31] E.P. Dolgenko, *On singularity removal of analytic functions*, Uspehi math. nauk, v. 18, n. 4, 1963, 135-142 (in Russian).

[32] Yu.V. Dymchenko, *Equality of capacity and module of a condenser on a surface*, Zapiski nauchn. seminarov LOMI, v. 276, 2001, 112-133 (in Russian).

[33] J. Eells and B. Fuglede, *Harmonic Maps between Riemannian Polyhedra*, Cambridge Tracts in Mathematics **142**, Cambridge Univ. Press, UK, 2001.

[34] A. Eremenko and D.H. Hamilton, *On the area distortion by quasiconformal mappings*, Proceedings of the American Mathematical Sosiety, **123**, n. 9, 1995, 2793-2797.

[35] M.A. Evgrafov, *Analytic functions*, Nauka, Moscow, 1968 (in Russian).

[36] D. Faraco, *Beltrami operators and microstructure, Academic dissertation*, Depart. of Math., Faculty of Sci., University of Helsinki, Helsinki, 2002.

[37] H. Federer, *Geometric Measure Theory*, Springer-Verlag, Berlin, 1969.

[38] R. Finn, *Remarks relevant to minimal surfaces, and to surfaces of prescribed mean curvature*, J. Analyze Math., v. 14, 1965, 139-160.

[39] A.T. Fomenko, *Variational methods in the topology*, Nauka, Moscow, 1982 (in Russian).

[40] B. Fuglede, *Extremal length and functional completetion*, Acta Math., v. 98, n.3-4, 1957, 171-219.

[41] F.R. Gantmacher, *Theory of matrices*, Nauka, Moscow, 1967 (in Russian).

[42] F.W. Gehring, O. Lehto, *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. A I, **272**, 1959, 1-9.

[43] F. Gehring and E. Reich, *Area distortion under quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. AI **388**, 1966, 1-14.

[44] E. Giusti, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Boston-Basel-Stuttgart, 1984.

[45] S.K. Godunov, E.I. Romenskii, *Elements of mechanics and conservation laws*, Izd-vo "Nauchnaya kniga", Novosibirsk, 1998 (in Russian).

[46] V.M. Goldstein, Yu.G. Reshetnyak, *Introduction to theory of functions with generalized derivatives and quasiconformal maps*, Nauka, Moscow, 1983 (in Russian).

[47] G.M. Golusin, *Geometric theory of complex variable functions*, Nauka, Moscow, 1966 (in Russian).

[48] C. Grammatico, *A result on strong unique continuation for the Laplace operator*, Commun. in partial defferential equations, v. 22 (9&10), 1997, 1475-1491.

[49] W. Gresky, *Konforme Abbildungen der Oberfläche Eines Rektangulären Hexaeders auf die Kugeloberfläche*, Inaugural-Dissertatin zur Erlangung def Doktorwürde der Hohen Philosophischen Fakultät der Universität Leipzig, Weida i. Thür, 1928, 1-74.

[50] E.G. Grigoryeva, A.A. Klyachin, and V.M. Miklyukov, *Problem of Functional Extension and Space-Like Surfaces in Minkowski Space*, Journals of Analysis and its Applications, v. 21, n. 3, 2002, 719-752.

[51] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc., v. 36, n. 2, 1999, 135-249.

[52] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Boston-Basel-Berlin, Birkhäuser, 1999.

[53] I.M. Grudskii, *Structure of inner coordinates on combined Riemannian surfaces*, In "Differential, integral equations and complex analysis", Elista, 1986, 30-45 (in Russian).

[54] I.M. Grudskii, *Christoffel-Schwarz formula for polyhedral surfaces*, Dokl. AN SSSR, v. 307, n. 1, 1989, 15-17 (in Russian).

[55] V. Gutlyanskii, O. Martio, T. Sugawa and M. Vuorinen, *On the Degenerate Beltrami Equation*, Reports of the Department of Mathematics, University of Helsinki, Preprint 282, 2001, 32 pp.

[56] P. Hajlasz, *Sobolev Mappings, Co-Area Formula and Related Topics*, Proceedings on analysis and geometry, Sobolev Institute Press, Novosibirsk, 2000, 227-254.

[57] H. Haken, *Advanced Synergetics*, Springer - Verlag, Berlin Heidelberg New York Tokyo, 1983.

[58] S.Ya. Havinson, *On erasure of singularities*, Litovskii math. sb., III, n. 1, 1963, 271-287 (in Russian).

[59] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Clarendon Press, Oxford etc., 1993.

[60] J. Heinonen, P. Koskela, *Sobolev Mappings with Integrable Dilatations*, Arch. Rational Mach. Anal., v. 125, 1993, 81-97.

[61] I.O. Herzog, *Phragmén-Lidelöf theorems for second order quasilinear elliptic partial differential equations*, Proc. Amer. Math. Soc., v. 15, 1964, 721-728.

[62] D.P. Il'yutko, *Locally minimal nets in N-normed spaces*, Math. zametki, v. 74, n. 5, 2003, 656-668 (in Russian).

[63] O.Yu. Imanuvilov, *On Carleman estimates for hyperbolic equations*, Asymptotic Analysis, v. 32, 2002, 185-220.

[64] T. Iwaniec, G. Martin, *Geometric Function Theory and Nonlinear Analysis*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2001.

[65] O.V. Ivanov, G.D. Suvorov, *Full lattice of conformally invariant compactifications of a domain*, Naukova Dumka, Kiew, 1982 (in Russian).

[66] J.A. Jenkins, *Univalent functions and conformal mapping* , Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958.

[67] A.P. Karmazin, *Quasiisometries, theory of preends and metric structures of space domains*, Izd-vo Surgutskogo un-ta, Surgut, 2003 (in Russian).

[68] R. Kersner, A. Shishkov, *Instantaneous Shrinking of the Support of Energy Solutions*, J. of Math. Anal. and Appl., v. 198, 1996, 729-750.

[69] V.A. Klyachin, V.M. Miklyukov, *Tubes and bands in the space-time*, Izd-vo Volgogradsk. univ., Volgograd, 2004 (in Russian).

[70] A.P. Kopylov, *Stability of Classes of Mappings and Hölder Continuity of Higher Derivatives of Elliptic Solutions to Systems of Nonlinear Differential Equations*, Sib. Math. J., v. 43, n. 1, 2002, 68-82 (in Russian).

[71] P. Koskela, J. Malý, *Mappings of finite distortion: The zero set of the Jacobian*, Journal of the European Mathematical Society, v. 5, n. 2, 2003, 95-105.

[72] V.I. Kruglikov, *On existence and uniqueness of maps quasiconformal in mean*, in 'Metr. voprosy theor. func. i otobr.', n. IV, Naukova Dumka, Kiew, 1973, 123-147 (in Russian).

[73] V.I. Kruglikov, *Condenser capacities and space maps quasiconformal in mean*, Math. sb., v. 130 (172), n. 2, 1986, 185-206 (in Russian).

[74] V.I. Kruglikov, V.M. Miklyukov, *Stability theorems for maps of the class BL*, in 'Metr. voprosy theor. functii i otobr.', n. III, Izd-vo Naukova Dumka, Kiew, 1971, 55-70 (in Russian).

[75] V.I. Kruglikov, V.M. Miklyukov, *On some classes of plane topological maps with generalized derivatives*, in 'Metr. voprosy theor. functii i otobr.', n. IV, Izd-vo Naukova Dumka, Kiew, 1973, 102-122 (in Russian).

[76] S.L. Krushkal', *On mappings quasiconformal in mean*, Sib. math. j., v. 8, n. 4, 1967, 798-806 (in Russian).

[77] S.L. Krushkal', *Quasiconformal mappings and Riemann surfaces*, John Wiley&Sons, New York-Toronto-London-Sydney, 1979.

[78] N. Kuiper, *On C^1-isometric embeddings*, 1, 2, Proc. Koninkl. Nederl. Akad. Wetensch. Ser. A, v. 58, n. 4, 5, 1955, 545-556, 683-689.

[79] B.P. Kufarev, *Potentials and boundary correspondence*, Izv. AN SSSR, Ser. math., v. 41, n. 2, 1977, 438-461 (in Russian).

[80] R. Kühnau, *Trianguliere Riemannsche Mannigfaltigkeiten mit ganz-linearen Bezugssub-Stitutionen and st"uchweise konstanter komplexer Dilatation*, Math. Nachrichten, B. 46, Heft 1-6, 1970, 243-261.

[81] R. Kühnau, *Über zwei klassen schlichter konformer Abbildungen*, Math. Nachr., B. 49, N. 1-6, 1971, 173-185.

[82] K. Kuratowski, *Topology*, v. I, Academic press, Warszawa, 1966.

[83] K. Lancaster, *Nonparametric minimal surfaces in R^3 whose boundaries have a jump discontinuity*, Internat. J. Math. Math. Sci., v. 11, n. 4, 1988, 651-656.

[84] M. Lavrentieff, *Sur une classe de representations continue*, Math. sb., v. 42, n. 4, 1935, 407-424.

[85] M.A. Lavrentiev, *On continuity of univalence functions in closed domains*, Dokl. AN SSSR, v. 4, 1936, 207-210 (in Russian).

[86] M.A. Lavrentiev, B.V. Shabat, *Problems of hydrodinamics and their mathematical models*, Nauka, Moscow, 1973 (in Russian).

[87] J. Lawrynowicz, *On a class of quasiconformal mappings with invariant boundary points, II, Applications and generalizations*, Ann. polon. math., v. 21, n. 3, 1969.

[88] N.A. Lebedev, *Area principle in the theory of univalent functions*, Nauka, Moscow, 1975 (in Russian).

[89] H. Lebesgue, *Sur le probleme de Dirichlét*, Rend. circ. mat. Palermo, **24**, 1907, 371-402.

[90] O. Lehto and K.I. Virtanen, *Quasiconformal Mappings in the Plane*, Second Edition, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[91] J. Lelong-Ferrand, *Représentation conforme et transformations a intégrale de Dirichlet bornée*, Gauthier-Villars, Paris, 1955.

[92] B.E. Levickii, V.M. Miklyukov, *Reduced Modulus on a Surface*, Vestnik Tomsk. Gosud. Univ., n. 301, Tomsk, 2007, 87-91 (in Russian).

[93] B.E. Levickii, I.P. Mityuk, *'Narrow' theorems on space module*, Dokl. AN SSSR, v. 248, n. 4, 1979, 780-783 (in Russian).

[94] V.P. Luferenko, G.D. Suvorov, *On concept of a prime end body in the Caratheodory theory*, in 'Metr. voprosy theor. func. i otobr.', 'Metr. vopr. teor. func. i otobr.', n. III, Naukova dumka, Kiew, 1971, 71-79 (in Russian).

[95] Y. Li and L. Nirenberg, *The Distance Function to the Boundary, Finsler Geometry, and the Singular Set of Viscosity Solutions of Some Hamilton-Jacobi Equations*, Communications of Pure and Applied Mathematics, v. LVIII, 2005, 85-146.

[96] P. Li and J. Wang, *Finiteness of disjoint minimal graphs*, Math. Res. Letters, v. 8, 2001, 771-777.

[97] J. Maly, *Sufficient Conditions for Change of Variables in Integral*, in 'Proc. on Analysis and geometry', Izd-vo Inst. of math., Novosibirsk, 2000, 370-386.

[98] O. Martio, V. Miklyukov, *On existence and uniqueness of degenerate Beltrami equation*, Complex Variables, v. 49, 2004, 647-656.

[99] O. Martio, V.M. Miklyukov, and M. Vuorinen, *Critical points of $A-$solutions of quasilinear elliptic equations*, Houston Journal of Mathematics, v. 25, n. 3, 1999, 583-601.

[100] O. Martio, V.M. Miklyukov, and M. Vuorinen, *Relative distance and boundary properties of nonparametric surfaces with finite area*, Journal of Mathematical Analysis and Applications, v. 286, n. 2, 2003, 524-539.

[101] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, *BMO-quasiconformal mappings and Q-homeomorphisms in space*, Prep. 288, Dept. Math. of Helsinki Univ., 2001, 24 pp.

[102] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov, *On the boundary behavior of Q-homeomorphisms*, Prep. 318, Dept. Math. of Helsinki Univ., 2002, 12 pp.

[103] V.G. Maz'ya, *On regularisity on the boundary of solutions of elliptic equations and conformal maps*, Dokl. AN SSSR, v. 152, n. 6, 1963, 1297-1300 (in Russian).

[104] E.J. McShane, *Parametrizations of saddle surfaces, with applications to the problem of Plateau*, Trans. Amer. Math. Soc., **35**, 1933, 716-733.

[105] D.E. Men'shov, *Les conditions de monogénéité*, Act. sci. et ind., 1936, v. 329, 1-52.

[106] V.M. Miklyukov, *On ε-quasiconformal mappings on a ball onto a ball*, in 'Metr. voprosy theor. func. i otobr.', n. I, Naukova Dumka, Kiew, 1969, 140-162 (in Russian).

[107] V.M. Miklyukov, *Some estimates of a conformal mapping of a domain to a strip*, Dokl. AN SSSR, v. 223, n. 3, 1975, 295-297 (in Russian).

[108] V.M. Miklyukov, *On some boundary problems of the conformal mapping theory*, Sib. math. j., v. 18, n. 5, 1977, 1111-1124 (in Russian).

[109] V.M. Miklyukov, *Two theorems on boundary properties of a minimal surface in the nonparametric form*, Math. zametki, v. 21, n. 4, 1977, 551-556 (in Russian).

[110] V.M. Miklyukov, *On the conformal type of surfaces, Liouville's theorem and Bernstein's theorem*, Sov. Math. Dokl., v. 19, n. 5, 1978, 1150-1153.

[111] V.M. Miklyukov, *On a new approach to Bernstein's theorem and related questions for equations of minimal surface type*, Math. USSR Sb., v. 36, n. 2, 1980, 251-271.

[112] V.M. Miklyukov, *On the asymptotic properties of subsolutions of quasilinear equations of elliptic type and mappings with bounded distortion*, Math. USSR Sb., v. 39, 1981, 37-60.

[113] V.M. Miklyukov, *Some singularities in the behavior of solutions of equations of minimal-surface type in unbounded domains*, Math. USSR sb., v. 44, n. 1, 1983, 61-73.

[114] V.M. Miklyukov, *Some questions of the qualitative theory of minimal surface type equations*, in 'Boundary problems of the mathematical physics', Naukova Dumka, Kiew, 1983 (in Russian).

[115] V.M. Miklyukov, *Remarks on the lenght and area principle*, in the G.D. Suvorov book 'Generalized lenght and area principle in the theory of maps', Naukova Dumka, Kiew, 1985, 252-260 (in Russian).

[116] V.M. Miklyukov, *On critical points of solutions of maximal surface type equations in the Minkowski space*, Teoriya otobr. i pribl. func., Naukova Dumka, Kiew, 1989, 112-125 (in Russian).

[117] V.M. Miklyukov, *Stagnation zones of solutions to the Laplace-Beltrami equation in long strips*, Siberian Advances in Mathematics, v. 12. n. 4. 2002. 1-17.

[118] V.M. Miklyukov, *Isothermic coordinates on singular surfaces*, Sbornik: Mathematics, v. 195, n. 1, 2004, 65-84.

[119] V.M. Miklyukov, *Relative Lavrent'eff Distance and Prime Ends on Non-parametric Surfaces*, Ukrainian Mathematical Bulletin, v. 1, n. 3, 2004, 353-376.

[120] V.M. Miklyukov, *On some applications of the M.A. Lavrentiev relative distance*, Dokl. RAN, v. 402, n. 4, 2005, 448-451 (in Russian).

[121] V.M. Miklyukov, *On a property of the height function of multi dimensional Plateau problem*, Math. i prikladn. anal., n. 2, Izd-vo Tyumenskogo univ., Tyumen', 2005, 110-119 (in Russian).

[122] V.M. Miklyukov, *Area distortion under conformal maps of surfaces*, Trudy seminara po vectorn. i tenzorn. anal., n. XXVI, Izd-vo MGU, Moscow, 2005, 238-249 (in Russian).

[123] V.M. Miklyukov, *Speed of Approximation to Degenerate Quasiconformal Mappings and Stability Problems*, Complex Analysis and Potential Theory, Proceedings of the Conference Satellite to ICM 2006, World Scientific Publishing Co. Pte. Ltd., 2007, 33-45.

[124] V.M. Miklyukov, S.S. Polupanov, R.A. Tarapata, *On stabilization of gas dinamics equation solutions*, Dokl. RAN, v. 388, n. 5, 2003, 596-598 (in Russian).

[125] V.M. Miklyukov, G.D. Suvorov, *On existence and uniqueness of quasiconformal maps with unbounded characteristics*, in 'Issled. po sovrem. teorii funct. and its appl.', Naukova Dumka, Kiew, 1972, 45-53 (in Russian).

[126] J. Milnor, *On deciding whether a surface is parabolic or hyperbolic*, Amer. Math. Monthly, v. 84, n. 1, 1977, 43-46.

[127] Ch.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, W.H.Freeman and Company, San Francisco, 1973.

[128] I.P. Mityuk, *Generalized reduced module and some of its applications*, Izv. vusov, Mathem., n. 2, 1964, 110-119 (in Russian).

[129] I.P. Mityuk, *On quasiconformal maps in the space*, Dopov. AN USSR, n. 8, 1964, 563-566 (in Ukrainian).

[130] V.N. Monachov, *Boundary problems with free boundaries for elliptic systems equations*, Nauka, Novosibirsk, 1977 (in Russian).

[131] M. Morse, *Topological methods in the theory of functions of a complex variables*, Princeton Univ. Press, Princeton, 1947.

[132] F. Morgan, *Geometric Measure Theory. A Beginners's Guide*, Second Edition, San Diego-Boston ets, Academic Press, 1995.

[133] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., v. 43, 1938, 126-186.

[134] S. Müller, V. Sverák, *On surfaces of finite total curvature*, J. Differential Geometry, v. 42, 1995, 229-258.

[135] J. Nash, C^1*-isometric imbeddings*, Ann. Math., v. 60, 1954, 383-396.

[136] S.R. Nasyrov, *The metric space of Rimann surfaces over the sphere*, Russian Acad. Sci. Sb. Math., v. 82, n. 2, 1995, 337-356.

[137] J.C.C. Nitsche, *On new results in the theory of minimal surfaces*, Bull. Amer. Math. Soc., v. 71, n. 2, 1965, 195-270.

[138] O.A. Oleinik, G.A. Iosif'yan, *Sent-Venan principle analog and uniqueness of solutions of boundary problems in unbounded domains for parabolic equations*, Uspehi math. nauk, v. 31, n. 6, 1976, 142-166 (in Russian).

[139] O.A. Oleinik, G.A. Iosif'yan, *Boundary value problems for second order elliptic equations in unbounded domains and Saint-Venant's Principle*, Ann. Suola Norm. Sup. Pisa Cl. Sci. (4), n. 2, 1977, 269-290.

[140] O.A. Oleinik, E.V. Radkevich, *Analyticity and Liouville and Phragmén-Lindelöf type theorems for general elliptic system equations*, Math. sb., v. 95 (137), n. 1, 1974, 130-145 (in Russian).

[141] R. Osserman, *Minimal surfaces*, Uspehi Math. Nauk, v. 22, n. 4, 1967, 55-135 (in Russian).

[142] Y. Pan and T. Wolf, *A Remark on Unique Continuation*, The J. of Geometric Analysis, v. 8, n. 4, 1998, 599-604.

[143] V.L. Panteleev, *Theory of Earth Figure*, Izd-vo MGU, Moscow, 2000, 160 pp. (in Russian)

[144] V.I. Pelich, *Phragmén-Lindelöf theorems on minimal surfaces*, in 'Geometr. anal. i ego prilog.', Izd-vo Volgograd. univ., volgograd, 1999, 352-368 (in Russian).

[145] G. Piranian, *The distribution of prime ends*, Mich. Math. J., v. 7, 1960, 83-95.

[146] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[147] I.I. Privalov, *Boundary properties of analytic functions*, GITTL, Moscov, 1950 (in Russian).

[148] Yu.G. Reshetnyak, *Stability theorems in geometry and analysis*, Nauka, Novosibirsk, 1982 (in Russian).

[149] Yu.G. Reshetnyak, *Two-dimensional manifolds of Bounded curvature*, in 'Geometry IV', Non-regular Riemannian Geometry, Springer-Verlag, Berlin-Heidelberg ets, 1993, 6-163.

[150] R.T. Rockafellar, *Convex analysis*, Princeton Univ. Press, Princeton, 1970.

[151] H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin-Göttingen ets, 1959.

[152] C.A. Rogers, *Hausdorff measures*, Cambridge at the University Press, 1970.

[153] B. Rodin, S.E. Warschawski, *Extremal length and the boundary behavior of conformal mapping*, Ann. Acad. Sci. Fenn. Ser. AI Math, v. 2, 1976, 467-500.

[154] B. Rodin, S.E. Warschawski, *Extremal length and univalent functions II. Integral estimates of strip mappings*, J. Math. Soc. Japan, v. 31, 1979, 87-99.

[155] B. Rodin, S.E. Warschawski, *Estimates for conformal maps of strip domains without boundary regularity*, Proc. London Math. Soc., v. 39, 1979, 356-384.

[156] B. Rodin, S.E. Warschawski, *On the derivative of the Riemann mappings function near a boundary point and the Visser-Ostrowski problem*, Math. Ann., v. 248, 1980, 125-137.

[157] B. Rodin, S.E. Warschawski, *A necessary and sufficient condition for the asymptotic version of Ahlfors' distortion property*, Trans. Amer. Math. Soc., v. 276, 1983, 281-288.

[158] B. Rodin, S.E. Warschawski, *Conformal mapping and locally asymptotically conformal curves*, Proc. London Math. Soc., v. 49, 1984, 255-273.

[159] V.I. Ryazanov, *On compactness and metrizability of kernel spaces of manifolds*, Ukrainian Math. J., v. 50, n. 6, 1999, 944-951 (in Russian).

[160] V. Ryazanov, U. Srebro, E. Yakubov, *Degenerate Beltrami equation and radial Q-homeomorphisms*, Preprint 369, August 2003, Department of Mathematics, University of Helsinki.

[161] S. Saks, *Theory of the integral*, Warsaw, 1939.

[162] L. Sario, M. Nakai, C. Wang, L.O. Chung, *Classification Theory of Riemannian Manifolds*, Lectures Notes Math., 1977, 605 pp.

[163] M. Schiffer and D.C. Spencer, *Functionals of finite Riemann surfaces*, Princeton University Press, Princeton, New Jersey, 1954.

[164] L. Schwartz, *Analyse mathématique I*, Hermann, 1967.

[165] H.A. Schwarz, *Conforme Abbildung der Oberfläche eines Tetraeders auf die Oberfläche einer Kugel.* - J reine angew. Math., v. 70, 1869, 121-136.

[166] B.V. Shabat, *Introduction to the complex analysis*, Nauka, Moscow, 1969 (in Russian).

[167] E.V. Shikin, *On parabolicity of immersiability and hyperbolicity of non-immersiability twodimensional manifolds of the negative curvature*, Vestn. Moscow un-ta. Mathem. Mech., n. 5, 1990, 42-45 (in Russian).

[168] A.E. Shishkov, *Behaviour of generalized solutions of the Dirichlét problem for general quasilinear divergence elliptic equations at the boundary neighborhood*, Different. equat., v. 23, n. 2, 1987, 308-320 (in Russian).

[169] J. Serrin, M. Tang, *Uniqueness of Ground States for Quasilinear Elliptic Equation*, Indiana Univ. Math. J., v. 49, n. 3, 2000, 897-923.

[170] L. Simon, *Equation of mean curvature in 2 independent variables*, Pacific J. Math., v. 69, n. 1, 1977, 245-268.

[171] V.I. Smirnov, *Kurs vysshei matematiki*, v. 5, Fismatgiz, Moscow, 1959 (in Russian).

[172] S.L. Sobolev, *Some applications of the functional analysis in the mathematical physics*, Izd-vo Leningrad. univ., Leningrad, 1950 (in Russian).

[173] J. Spruck, *Two dimensional graphs over unbounded domains*, J. Inst. Math. Jussien, v. 1, 2002, 631-640.

[174] S. Stoilow, *Le cons sur les principes topologiques de la theorie de fonctions analytiques*, Cauthier - Villars, Paris, 1938.

[175] Yu.F. Strugov, *Quaiconformal in mean maps and extremal problems*, Parth 1, Omsk, 1994, 154 pp. - Dep. in VINITI 05.12.94, No 2786-94; Parth 2, Omsk, 1994, 114 pp. - Dep. in VINITI 05.12.94, No 2787-94 (in Russian).

[176] G.D. Suvorov, *Remarks to a M.A. Lavrentiev theorem*, Uch. zap. Tomsk. univ., v. 25, 1956, 3-8 (in Russian).

[177] G.D. Suvorov, *Families of plane topological maps*, Izd-vo SO AN SSSR, Novosibirsk, 1965 (in Russian).

[178] G.D. Suvorov, *Uniform stability of conformal maps of closed domans*, Ukrain. math. j., v. 20, n. 1, 1968, 78-84 (in Russian).

[179] G.D. Suvorov, *Generalized 'length and area' principle in the theory of maps*, Naukova Dumka, Kiew, 1985 (in Russian).

[180] G.D. Suvorov, *Prime ends and sequences of plane domains of plane maps*, Naukova Dumka, Kiew, 1986 (in Russian).

[181] A.V. Sychev, *Module and space quasiconformal maps*, Nauka, Novosibirsk, 1983 (in Russian).

[182] A.F. Tedeev, *Qualitative properties of solutions of the Neyman problem for a quasilinear parabolic equation of the height order*, Ukr. math. j., v. 45, n. 11, 1993, 1571-1579 (in Russian).

[183] O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbildung*, Dtsch. Math., n. 3, 1938, 621-678.

[184] V.G. Tkachev, A.N. Ushakov, *Fuglede theorem in Finsler spaces*, in 'The potential theory', Naukova Dumka, Kiew, 1991, p. 15 (in Russian).

[185] T. Toro, *Surfaces with generalized second fundamental form in L^2 are Lipschitz manifolds*, J. Differential Geometry, v. 39, 1994, 65-101.

[186] Yu.Yu. Trohimchyk, *Continuous maps and conditions of monogeneity*, GFML, Moscow, 1963 (in Russian).

[187] D.C. Ullrich, *Removable sets for harmonic functions*, Mich. Math. J., 1991, v. 38, n. 3, 467-473.

[188] N.X. Uy, *Removable set for Lipschitz harmonic functions*, Mich. Math. J., v. 37, 1990, 45-51.

[189] I.N. Vekua, *Generalized analytic functions*, GIFML, Moscow, 1959 (in Russian).

[190] G.M. Vergbickii, V.G. Maz'ya, *Asymptotic behaviour of second order elliptic equations close to boundary, I*, Sib. math. j., v. 12, n. 6, 1971, 1217-1249.

[191] G.M. Vergbickii, V.G. Maz'ya, *Asymptotic behaviour of second order elliptic equations close to boundary, II*, Sib. math. j., v. 13, n. 6, 1972, 1239-1271.

[192] L.I. Volkovyskii, *Quasiconformal mappings*, Izd-vo L'vov univ., L'vov, 1954 (in Russian).

[193] I.A. Volynec, *On distortion under mappings of the class BL*, Sib. math. j., v. 18, n. 6, 1977, 1259-1270 (in Russian).

[194] A.G. Vorob'ev, V.M. Miklyukov, *On some properties of subsolutions of minimal surface type equations* , Sib. math. j., v. 23, n. 1, 1982, 25-31.

[195] S.E. Warschawski, *Über das Randverhalten der Ableitung der Abbildungsfunction bei konformer Abbildung*, Math. Z., v. 35, 1932, 321-456.

[196] S.E. Warschawski, *On conformal mapping of infinite strips*, Trans. Amer. Math. Soc., v. 51, 1942, 280-335.

[197] S.E. Warschawski, *On the boundary behaviour of conformal mappings*, Nagoya Math. J., v. 30, 1967, 83-100.

[198] A. Weitsman, *On the growth of minimal graphs*, Indiana Univ. Math. J., v. 54(2), 2005, 617-625.

[199] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc., v. 36, 1934, 63-89.

[200] T.H. Wolff, *Note on counterexamples in strong unique continuation problems*, Proc. of the Amer. Math. Soc., v. 114, n. 2, 1992, 351-356.

[201] T.H. Wolff, *A counterexample in a unique continuation problem*, Comm. in Analysis and Geometry, v. 2, n. 1, 1994, 79-102.

[202] V.A. Zhukov, *On a proof of the theorem on differential existence for W_p^1-functions*, Trudy Tomsk. un-ta, Ser. mech.-math., v. 189, 1966, 13-17 (in Russian).